Cauchy-Green 公式

定理 1. 设区域 $D\subset\mathbb{C}$，边界是分段 C^1 的曲线. 设 $\mathit{\Omega}$ 是 $\overline{D}$ 的开邻域，$\gamma=\partial D$，$f\in C^1(\mathit{\Omega})$，则

$$\int_\gamma f(z)\,\mathrm{d}z=\int_D\frac{\partial f}{\partial\overline{z}}\,\mathrm{d}\overline{z}\wedge\mathrm{d}z.$$

证明. 设 $f(z)=f(x+\mathrm{i}y)=u+\mathrm{i}v$，由 Green 公式及 $\mathrm{d}\overline{z}\wedge\mathrm{d}z=2\mathrm{i}\,\mathrm{d}x\wedge\mathrm{d}y$，有

$$\begin{aligned}\int_{\partial D}f(z)\,\mathrm{d}z&=\int_{\partial D}(u\,\mathrm{d}x-v\,\mathrm{d}y)+\mathrm{i}\int_{\partial D}(v\,\mathrm{d}x+u\,\mathrm{d}y)\\&=\int_D(-v_x-u_y)\,\mathrm{d}x\wedge\mathrm{d}y+\mathrm{i}\int_D(u_x-v_y)\,\mathrm{d}x\wedge\mathrm{d}y\\&=\int_D[(u_x-v_y)+\mathrm{i}(v_x+u_y)]\,\mathrm{d}\overline{z}\wedge\mathrm{d}z\\&=\int_D\frac{\partial f}{\partial\overline{z}}\,\mathrm{d}\overline{z}\wedge\mathrm{d}z.\end{aligned}$$

□

推论 1 (Cauchy 积分定理)**.** 设区域 $D\subset\mathbb{C}$，边界是分段 C^1 的曲线. 设 $\mathit{\Omega}$ 是 $\overline{D}$ 的开邻域，$\gamma=\partial D$，$f\in C^1(\mathit{\Omega})$，若 $f(x)$ 还是全纯函数，即 $f\in H(\mathit{\Omega})$，则

$$\int_\gamma f(z)\,\mathrm{d}z=0.$$

定理 2 (Cauchy-Green,Pompeiu)**.** 设区域 $D\subset\mathbb{C}$，边界是分段 C^1 的曲线. 设 $\mathit{\Omega}$ 是 $\overline{D}$ 的开邻域，$f\in C^1(\mathit{\Omega})$，对 $z\in D$，有

$$f(z)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta-\frac{1}{2\pi\mathrm{i}}\int_D\frac{\partial f(\zeta)}{\partial\overline{\zeta}}\cdot\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{\zeta-z}.$$

证明. 取 $\varepsilon>0$ 使 $\overline{B}(z,\varepsilon)\subseteq D$. 记 $D_{z,\varepsilon}=D\backslash\overline{B}(z,\varepsilon)$，当 ε 充分小时总有 $D_{z,\varepsilon}$ 是连通的. 在 $D_{z,\varepsilon}$ 上对

$$F(z)=\frac{f(\zeta)}{\zeta-z}$$

关于 ζ 积分，由 Green 公式，有

$$\left(\int_\gamma-\int_{\gamma_{z,\varepsilon}}\right)\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta=\int_{D_{z,\varepsilon}}\frac{\partial f(\zeta)}{\partial\overline{\zeta}}\cdot\frac{\mathrm{d}\overline{\zeta}\wedge\mathrm{d}\zeta}{\zeta-z}.\tag{1}$$

因为 $f\in C^1$，所以存在常数 M 使得任意 $z_1,z_2\in\overline{B}(z,\varepsilon)$，有

$$|f(z_1)-f(z_2)|\leqslant M|z_1-z_2|.$$

于是对于 $\zeta \in \gamma_{z,\varepsilon}$ 有

$$|f(\zeta) - f(z)| < M|\zeta - z|.$$

所以

$$\left| \int_{\gamma_{z,\varepsilon}} \frac{f(\zeta) - f(z)}{\zeta - z} \, \mathrm{d}\zeta \right| \leqslant \int_{\gamma_{z,\varepsilon}} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| |\, \mathrm{d}\zeta| \leqslant M \cdot 2\pi\varepsilon.$$

令 $\zeta = z + r\mathrm{e}^{\mathrm{i}\theta}$，$r \in [0, \varepsilon]$，$\theta \in [0, 2\pi]$，则存在函数 $G(r, \theta)$ 及常数 M_G，使得

$$\left| \int_{B(z,\varepsilon)} \frac{\partial f(\zeta)}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \delta\zeta}{\zeta - z} \right| = \left| \int_0^r \int_0^{2\pi} G(r, \theta) \mathrm{e}^{-i\theta} \, \mathrm{d}r \wedge \mathrm{d}\theta \right| \leqslant M_G \pi \varepsilon^2.$$

而方程(1)等价于

$$\int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \int_{\gamma_{z,\varepsilon}} \frac{f(\zeta) - f(z)}{\zeta - z} \, \mathrm{d}\zeta - \int_{\gamma_{z,\varepsilon}} \frac{f(z)}{\zeta - z} \, \mathrm{d}\zeta = \left(\int_D - \int_{B(z,\varepsilon)} \right) \frac{\partial f(\zeta)}{\partial \overline{\zeta}} \cdot \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{\zeta - z}.$$

令 ε 趋于 0 即可. □

推论 2 (Cauchy 积分公式)**.** 设区域 $D \subset \mathbb{C}$，边界是分段 C^1 的曲线. 设 $\mathit{\Omega}$ 是 $\overline{D}$ 的开邻域，$f \in C^1(\mathit{\Omega}) \cap H(\mathit{\Omega})$，对 $z \in D$，有

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

证明. 由 C-R 方程，有

$$\frac{\partial f(\zeta)}{\partial \overline{\zeta}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + \mathrm{i} \frac{\partial f}{\partial y} \right) = 0.$$

□

由 Cauchy 积分公式也可以推出 Cauchy 积分定理，因此二者是相互等价的. 只需取 $F(\zeta) = (\zeta - z) f(\zeta)$，则 $F(z) = 0$，对 F 应用 Cauchy 积分公式，有

$$0 = F(z) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{F(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \frac{1}{2\pi\mathrm{i}} \int_\gamma f(\zeta) \, \mathrm{d}\zeta.$$